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Clifford Algebra Example Science Page

- 1. Introduction
  - 1
- 2. Expanded
  - 2

1. Introduction

1

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#cliffordalgebra - A short summary of Clifford Algebra-based methodology as a unified alternative to tensor calculus in classical, relativistic, quantum physics and electromagnetics

1/ Clifford Algebra. Cl3,0(ℝ) can be constructed of a triple base (referred to as 'primary') of anti-commuting operators e,f,g (corresponding to y,z,x in this order). A 'product' of two bases (operators) is interpreted as a superposition of operators. A superposition of the same operator (with itself) gives identity I, that is: ee=ff= gg=I. Superposition of different operators produces 'bivector' bases (also referred to as 'secondary' base) i,j,k that is ef=-fe=i, fg=-gf=j, ge=-eg=k. Note that any and all linear operators in Cl3,0(ℝ) need to be defined only on the three primary bases {e,f,g}, and can be extended on the entire Cl3,0(ℝ) domain using the axioms of the algebra.

2/ Trivector. Trivector is a triple product of base vectors (operators) efg=i . It is commutative with all other base operators (base vectors) and has the same property as the imaginary unit of the Complex number field, that is ii = -I. Full vector base consists of {I,e,f,g,i,j,k,i} . A general Clifford vector can be constructed the same way as in classical Cartesian (or more generally in Hilbert space) vector algebra by linear combination of its base vectors with real coefficients (though it is important to keep in mind that unlike in the former, Clifford base vectors are not generally orthogonal).

3/ Scalar and Imaginary Scalar . Identity and trivector bases {I,i} have the same algebraic properties as the real number 1 (we will write 1 instead of I) and imaginary number i . Thus the sub-algebra {1,i} is isomorphic to Complex number field ℂ allowing self-complexification of Cl3,0(ℝ) = {1,e,f,g,i,j,k,i}xℝ into an equivalent complex version of this algebra {1,e,f,g}x{1,h} or {1,e,f,g}xℂ. [Note]

4/ Full Clifford 8-vectors . Full Clifford vectors represent physical objects called Spinors. Wave-function of a fermion particle is a spinor. A Complex 4-vector **a**= a_0+a_1e+a_2f+a_3g (where a_0,a_1,a_2,a_3 are arbitrary Complex coefficients) . a represents a Dirac spinor. If coefficients a_0,a_1,a_2,a_3 are Real then **a** represents Einstein/Minkowski 4-vector. If a_0,a_1,a_2,a_3 have a common complex scale factor of the type exp(i□□) which can be factored out of each a_i = exp(i□□)b_i where b_i is real, i=0..3, then □□ can be interpreted as the quantum phase of boson particle **a**.

2. Expanded

2. Expanded

2

5/ Relativistic 4-vectors. Real Clifford 4-vectors can be made to represent the classical and relativistic kinematic vectors for example path increment $d\square = dt+dx_1e+dx_2f+dx_3g$, energy+momentum 4-vector, electromagnetic potential 4-vector or charge-density + current density 4-vector etc. Complex Clifford vectors represent wave functions. The Clifford product of two wave-function vectors is interpreted as sequential superposition of two quantum events, representing evolution of quantum amplitude of physical processes taking place sequentially. Vector addition of two wave-function vectors represents the combined quantum amplitude of two probable quantum processes taking place simultaneously along different quantum trajectories each, in its parallel multiverse, as per Feynman's diagram model.

Reference: "Standard model physics from an algebra?" by Cohl Furey, PhD thesis, Waterloo, Ontario, Canada, 2015 https://arxiv.org/pdf/1611.09182.pdf... Note: Dr. Furey employs Octonions Cl(6)xC, equivalent to Cl7,0(ℝ) based on the notation I am using. Cl3,0(ℝ) is isomorphic to Complex Quaternions

6/ Exponential function. Clifford vectors can be exponentiated. Exponent function is defined as the infinite sum

exp(v) := 1+v+vv/2+vvv/6+...+vn/n!+...

Important class of arguments are vectors that are linear combinations of bases {e,f,g} only, with all Real or all Imaginary coefficients (without the unity scalar term). Such vectors are called transformation 'generator' vectors for the general transformation formula of the relativistic 4-vectors from one coordinate system to another: a → a' :

a' = exp(v) a exp(vT)

Note: Operator $(\ )^T$ is called Hermitian Conjugate, defined as an involution function Cl3,0(ℝ) → Cl3,0(ℝ) that reverses the order of multiplication of two Clifford vectors (that is, for all a,b : (ab)T = bTaT) but does not alter the signs of {e,f,g}. Example: $e^T=e$, $1^T=1$, $i^T=-i$, $i^T=(ef)^T = -fe = -i$.

7/ Rotation and Lorentz transform of the 4-vectors.

General transform. The following formula

$a'=\exp(v)\ a\ \exp(v^T)$

produces 3D rotation transform and relativistic (Lorentz) transform of 4-vectors of type a,b = {1,e,f,g}x$\mathbb{R}$ with real coefficients only![ ] Minkowski's/Einstein's 4-vectors, for example a=(t,x,y,z) where t,x,y,z are all Real coordinate coefficients, can be represented in Clifford notation as a=t+xe+yf+zg . [ ]

3D Rotation uses v={e,f,g}xImaginary, resulting in the transformation formula:

a'=exp(v) a exp(-v).

Notice that Clifford modulus of a defined as $|a|^2 = a\,a^{\sim}$ is an invariant of the Lorentz and rotation transforms [] :

$|a|^2 = t^2 - x^2 - y^2 - z^2$

Note: Operator ( )~ is called Clifford Conjugation or Quaternion Conjugation defined as an involution that flips the signs of bases {e,f,g} and reverses the multiplication order. That is, for all a,b : (ab)~ = b~a~ and it toggles the signs of {e,f,g}. Example: e~=-e , 1~=1, i~=i, i~=(ef)~ = -fe = -i .

Example of rotation

Let us use the following transform generator vector v:=i□□g : a'=exp(i□□g) a exp(-i□□g), a=t+xe+yf+zg. We need to re-group the terms separating those that are commutative with g, and those that are anti-commutative with g: [ ]

a' = exp(i□□g)(t+zg)exp(-i□□g) + exp(i□□g)(xe+yf)exp(-i□□g) =

(t+zg) + exp(2i□□g)(xe+yf) =

(x cos2□□ + y sin2□□)e +

(-x sin2□□ + y cos2□□)f + t + zg

The first two terms at e and f are identical to a rotation matrix by angle 2□□ along the z (g) axis. The scalar t and z components are of course rotationally invariants, in this case.